Theoretical Computer Science 17 (1982) 29–41
North-Holland Publishing Company

COLOR-FAMILIES ARE DENSE

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Communicated by A. Salomaa
Received July 1980
Revised September 1980

Abstract. Graphs, regarded as grammar forms as well as coloring specifications, induce graph-families, so-called color-families. In this paper a minimal producer-graph for every color-family is introduced and as the main result it is shown that color-families are dense, in the sense that between any two families one can 'squeeze in' another one.

1. Introduction

The investigation of grammar forms offers—in the case of languages containing only words of length two—a link to graphs, first observed in [2]. Thus, the mechanism of interpretation, applied to graphs, turns out to be a generalization of the notion of n-coloring and induces graph-families (we call them color-families) in a similar way, as it does for grammar forms. This paper is confined to undirected graphs. However, most of the results can be extended to digraphs.

In Section 2 the basic definitions and some results of [2] are given without proofs. Then in Section 3 a minimal representative graph for every color-family is introduced and some unary and binary operations on graphs, with their consequences on the properties of the graphs are investigated. Finally, in Sections 4 and 5 the main open question of [2], whether color-families are dense, is answered affirmatively.

The paper is largely self-contained. For proofs not given in Section 2 the reader is referred to [2], where the directed case is discussed, too (e.g. examples of non-density concerning digraphs are given). [3] and [4] investigate some expansions of the results in [2] and this paper on finite grammar forms. For unexplained notions in graph theory I refer to [1]. [5] is a general introduction to the theory of forms.

2. Preliminaries

In this section we review definitions and results from [2] as required for the rest of the paper.

Definition. Let G and G' be graphs. G' is termed an *interpretation* of G modulo μ, in symbols $G' \lhd G(\mu)$, if the following two conditions obtain:

(i) μ is a mapping of the set of vertices of G into the set of subsets of the set of vertices of G' such that

$$x_1 \neq x_2 \text{ implies } \mu(x_1) \cap \mu(x_2) = \emptyset$$

and, moreover, every vertex of G' belongs to one of the sets $\mu(x)$;

(ii) whenever there is an edge between y_1 and y_2 in G', there is also an edge between $\mu^{-1}(y_1)$ and $\mu^{-1}(y_2)$ in G.

The above definition shows the close connection to grammar forms. However, when we speak of interpretation in the sequel, we mainly think in terms of the following theorem for whose formulation we need two more definitions.

An *elementary homomorphism* in a graph G consists of identifying two vertices x and y and inserting an edge between the identified vertex $x = y$ and all vertices z adjacent to either x or y in G. (When x and y are adjacent, $x = y$ has a loop—this means an edge from the vertex to itself). A graph G^* is a *morphic image* of a graph G if it is obtained from G by finitely many elementary homomorphisms. G is also considered to be a morphic image of itself.

Theorem 2.1. *A graph G is an interpretation of a graph H if and only if a morphic image G^* of G is* (*isomorphic to*) *a subgraph of H.*

Every graph G defines a family $\mathcal{L}(G)$ of graphs, consisting of all interpretations of G, in symbols

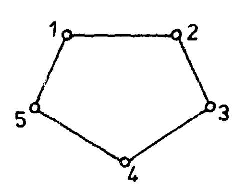
$$\mathcal{L}(G) = \{G' \mid G' \lhd G(\mu)\}.$$

The reader should have no difficulties to verify that $\mathcal{L}(U)$ (whereat U is a vertex with a loop) consists of all graphs and that no other graph without a loop can have this property.

In addition to a connection with grammar forms, we have a link to the theory of colorings as follows: A graph G is an interpretation of K_n (K_n is the complete graph with n vertices) iff G is n-colorable. Therefore, the following definition is a natural extension of the notion of coloring.

Definition. For two graphs G and H, G is *H-colorable* if $G \lhd H$. A family of graphs is a *color-family* if it equals $\mathcal{L}(H)$, for some graph H.

For instance, consider the cyclic graph C_5 (C_n is the cycle with n vertices)



A graph is C_5-colorable if and only if it is 5-colorable in such a way that the adjacencies of C_5 are satisfied: if a vertex is colored by 1, then its neighbors may be colored by 2 or 5 but not by 3 or 4, and so forth.

The following theorem is a rather direct consequence of the definitions.

Theorem 2.2. (i) *The relation 'interpretation of' is transitive.*

(ii) *The inclusion $\mathscr{L}(G_1) \subset \mathscr{L}(G_2)$ holds if and only if $G_1 \lhd G_2$.*

(iii) *The relation 'interpretation of' is decidable.*

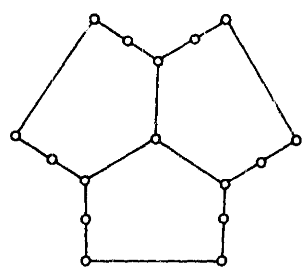
Now we are ready to introduce the basic hierarchy:

$$\mathscr{L}(K_2) \subsetneq \cdots \mathscr{L}(C_{2m+1}) \subsetneq \mathscr{L}(C_{2m-1}) \subsetneq \cdots \mathscr{L}(C_3) = \mathscr{L}(K_3)$$
$$\subsetneq \mathscr{L}(K_4) \subsetneq \cdots \mathscr{L}(K_n) \subsetneq \mathscr{L}(K_{n+1}) \cdots \tag{1}$$

The even cycles C_{2m} are omitted in the hierarchy, because all of them define the same family as K_2. In general a graph G is in $\mathscr{L}(K_2)$ iff G has no odd cycle as subgraph. Concerning the hierarchy (1) the question arises (especially from the point of view of grammar forms) whether there are color-families strictly between these C_{2m+1}'s and K_n's. The families defined by the so called 'flowers' satisfy this claim for the odd cycles.

Definition. The *flower* F^n_m is the planar graph obtained by gluing $2n+1$ copies of the graph C_{2m+1} together in the following fashion. All $2n+1$ copies possess a common vertex x. Two neighboring copies have also a vertex adjacent to x in common. Otherwise, the sets of vertices of any two copies are disjoint.

For instance F^1_3



Note that F^0_m equals C_{2m+1}.

Theorem 2.3. *For any $m \geq 1$, $n \geq 0$,*

$$\mathscr{L}(F^0_{m+1}) \subsetneq \mathscr{L}(F^{n+2}_{m+1}) \subsetneq \mathscr{L}(F^{n+1}_{m+1}) \subsetneq \mathscr{L}(F^0_m)$$

holds.

The odd girth of a graph G, $\mathrm{og}(G)$, is the length of the shortest odd cycle in G (clearly, if there is a loop in G we have $\mathrm{og}(G) = 1$ and by definition we say $\mathrm{og}(G) = \infty$ if no odd cycle exists). This graph property gives us the following simple but helpful

Lemma 2.4. *Let G and H be graphs, where $\mathrm{og}(G) > \mathrm{og}(H)$. Then H is not an interpretation of G.*

3. Minimal graphs and operations on graphs

Two graphs G and H are termed *form equivalent*, in symbols $G \sim H$, if they define the same graph-family, i.e. $\mathscr{L}(G) = \mathscr{L}(H)$. If G is an interpretation of H, but not form equivalent, we say G is a *strong interpretation* of H, in symbols $G \lneqq H$. In this section we consider a minimal form equivalent graph for every graph G.

Observation 3.1. (Immediate consequences of Theorem 2.1).

(i) G is an interpretation of its morphic images,

(ii) G is a subgraph of H implies that G is an interpretation of H,

(iii) every graph is form equivalent to its morphic subgraphs.

A graph M is a *minimal graph* when M has no morphic subgraph (except M itself). Of course, every graph has a morphic subgraph which is minimal graph (take morphic subgraphs as long as possible).

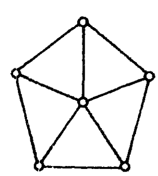
Lemma 3.2. *Every color-family has a uniquely defined minimal graph as representative.*

Proof. Suppose two form equivalent minimal graphs M and N are not isomorphic. As M is an interpretation of N there is a morphic image M^* of M which is a subgraph of N. N is an interpretation of M and therefore M^* is an interpretation of M. Hence, there exists a morphic image M^{**} of M^* which is a subgraph of M. As M^{**} is a morphic image of M we have $M^{**} = M^* = M$ by definition of minimal graph and consequently M is a subgraph of N. The analagous considerations for N show that N is a subgraph of M and therefore M is isomorphic to N.

Obviously, all complete graphs and odd cycles are minimal graphs. Further investigation shows that K_1, K_2, K_3, K_4, K_5 and C_5 are the only minimal graphs with five or less vertices.

Definition. The *1-enlarged* graph $M+1$ of M is obtained by adding a vertex x to M and inserting edges between x and all vertices of M. The *n-enlarged* graph of M is defined recursively: $M+n = (M+(n-1))+1$.

For instance: $C_5 + 1$:



Lemma 3.3. (i) *$M+n$ is a minimal graph if and only if M is a minimal graph (we assume that M is not a vertex with a loop).*

(ii) *If* $G \sim H$, *then* $G+n \sim H+n$.
(iii) *If* $G \trianglelefteq H$, *then* $G+n \trianglelefteq H+n$.

Proof. (i) Because of the recursive definition of $M+n$ we just have to show that $M+1$ is a minimal graph. Assume $M+1$ is not. Then there is a morphic image $M+1^*$ which is a subgraph of $M+1$. The added vertex x cannot be identified without producing a loop, hence we can say M^*+1 instead of $M+1^*$. When M^*+1 is a subgraph of $M+1$, there is an injective function i mapping the vertices of M^*+1 to those of $M+1$ such that $i(a)$ is adjacent to $i(b)$ if a and b are adjacent.

Let x be the added vertex in $M+1$ and x^* the added vertex in M^*+1.

Case 1: Assume that $i(x^*)=x$ or that there is no vertex y^* such that $i(y^*)=x$. This implies that M^* is a subgraph of M, which contradicts that M is a minimal graph.

Case 2: Assume that $i(x^*)\neq x$ and that there is a vertex y^* with $i(y^*)=x$. Let y be the vertex for which we have $i(x^*)=y$. But then

$$i': \qquad i'(x^*)=x, \qquad i'(y^*)=y \quad \text{and} \quad i'(z^*)=i(z^*) \quad \text{for } z^*\neq x^*, y^*$$

is also a valid mapping leading to a contradiction as in Case 1.

(ii) G^* subgraph of H for a morphic image G^* of G implies G^*+1 subgraph of $H+1$. Hence $G \lhd H$ implies $G+1 \lhd H+1$ and form equivalence is preserved, too.

(iii) Let $\min(G)$ be the minimal graph of G. Then

$$G \sim \min(G) \trianglelefteq \min(H) \sim H$$

implies

$$G+1 \sim \min(G)+1 \trianglelefteq \min(H)+1 \sim H+1$$

where the middle relation is strong because different minimal graphs cannot be form equivalent.

With the enlarge-operation we can transform the complete hierarchy of odd cycles and flowers between any arbitrary pair $K_n - K_{n+1}$ $(n \geq 3)$, which implies

Corollary 3.4. *For* $n \geq 2$ *there are infinitely many distinct color-families between* $\mathscr{L}(K_n)$ *and* $\mathscr{L}(K_{n+1})$.

$A \cup B$ is the graph obtained by the (disjoint) union of the graphs A and B. Due to [2] we have

Lemma 3.5. *Let F, G, H be graphs where H is connected and* $F \lhd H$, $G \trianglelefteq H$, $G \trianglelefteq F$. *Then the relation* $F \trianglelefteq F \cup G \trianglelefteq H$ *holds.*

Lemma 3.6. (i) *Every color-family is closed under union.*

(ii) $\mathscr{L}(G) \cup \mathscr{L}(H)$ *is a color-family if and only if* $\mathscr{L}(G) \subset \mathscr{L}(H)$ *or* $\mathscr{L}(G) \supset \mathscr{L}(H)$ *(i.e. G and H are comparable).*

Proof. (i) Consider a family $\mathscr{L}(G)$ and two graphs H_1 and H_2 in $\mathscr{L}(G)$. Then there are morphic images H_1^* and H_2^*, which are subgraphs of G. Hence, there is a morphic image $(H_1^* \cup H_2^*)^*$ which is a fusion of H_1^* and H_2^*, so that $(H_1^* \cup H_2^*)^*$ is a subgraph of G.

(ii) Consider two families $\mathscr{L}(G)$ and $\mathscr{L}(H)$ which are incomparable ($\bar{\mathscr{L}} = \mathscr{L}(G) \cup \mathscr{L}(H)$). Of course, H and G are in $\bar{\mathscr{L}}$. Suppose $\bar{\mathscr{L}}$ is a color-family. Then $G \cup H$ is in $\bar{\mathscr{L}}$. This is impossible because $G \cup H$ is neither in $\mathscr{L}(G)$ nor in $\mathscr{L}(H)$.

4. Predecessors and weak predecessors

Let P be a strong interpretation of S. P is called a *predecessor* of S, in symbols $P.\mathrm{p}.S$, (S a *successor* of P), if there is no graph G, so that the relation $P \lneqq G \lneqq S$ holds.

Lemma 4.1. *Let P be a predecessor of G, where G is connected. Then*
(i) *any other predecessor P' of G is form equivalent to P and*
(ii) *H is a strong interpretation of G implies H is an interpretation of P.*

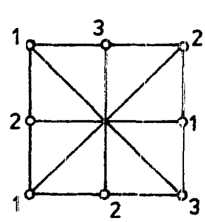
Proof. (i) Suppose P and P' are two non-equivalent predecessors of H. Of course, P and P' are incomparable (otherwise P or P' is not a predecessor of G). Since G is connected this implies $P \lneqq P \cup P' \lneqq G$ which is also a contradiction.

(ii) If H is a strong interpretation of G and not an interpretation of P, then H and P are incomparable and we have a similar contradiction as before with the relation $P \lneqq P \cup H \lneqq G$.

Note that this lemma does not show that G is the only successor of P.

Definition. W is called a *weak predecessor* of G, in symbols $W.\mathrm{wp}.G$, if $W \lneqq G$ and W^* is a strong interpretation of G does not hold for any morphic image W^* of W, except W itself.

In addition to $C_5.\mathrm{wp}.C_3$, we have another example for a weak predecessor as follows: The graph H:



is a strong interpretation of C_3 ('interpretation' shown by the coloring, 'strong', because there is no C_3 in H). Two vertices x and y in a graph are termed to be in C_{2m+1}-*connection* if a cycle C_{2m+1} which contains x and y is a subgraph of this graph.

One can easily verify that any pair of vertices in the graph H is in C_5-connection and therefore any elementary homomorphism on H produces a morphic image H^* with a C_3 (or a loop) as subgraph. Consequently, $H^* \lhd C_3$ does not hold. This implies $H.\mathrm{wp}.C_3$.

As we shall see by the following algorithm, every graph G which is a strong interpretation of H induces a morphic image of G which is a weak predecessor of H in a nondeterministic way.

Algorithm 4.2. (weak(G, H) produces a morphic image of G such that weak(G, H).wp.H holds, $G \lhd H$ being assumed).

> **graph procedure** weak(G, H)
> **if** (there is no elementary homomorphism h on G, so that
> $h(G) \lhd H$ holds) **then assign** weak $:= G$;
> **else** take an arbitrary el.hom., which assures the relation $h(G) \lhd H$ and
> **assign** weak $:=$ weak$(h(G), H)$;

Obviously, the algorithm is nondeterministic, but stops successfully after finitely many steps. It induces a sequence of elementary homomorphisms, so that weak$(G, H) = G^* = h_i h_{i-1} \cdots h_2 h_1(G)$; this sequence results in a homomorphism $h := h_i h_{i-1} \cdots h_2 h_1$.

Remark 4.3. If $h(x) = h(y)$ and h_1' is an elementary homomorphism, identifying x and y, then the relation $h_1'(G) \lhd H$ holds because there is another sequence $h_j' h_{j-1}' \cdots h_2' h_1' = h$ and we have $G \lhd h_1'(G) \lhd (h_j' h_{j-1}' \cdots h_2') h_1'(G) = h(G) \lhd H$. This fact, just mentioned, will turn out very important later on.

In the following lemma, the important interrelation between weak predecessors and predecessors is shown.

Lemma 4.4. *Let P be a minimal graph, G a connected graph and P.p.G. Then*
(i) *P is a weak predecessor of G and*
(ii) *every weak predecessor W of G is a subgraph of P.*

Proof. (i) P is a minimal graph and therefore P is a strong interpretation of any morphic image P^* (not equal to P) of P. Consequently, $P^* \lhd G$ would squeeze P^* strongly between P and G.

(ii) according to Lemma 4.1, we know $W \lhd P$ for every weak predecessor W. If W is not a subgraph of P, there must be a morphic image W^* which is a subgraph of P. The transitivity of interpretation implies immediately $W^* \lhd G$. This contradicts the definition of weak predecessor.

Above lemma is a helpful tool for proving that a graph G has no predecessor. We just have to show that G has weak predecessors of arbitrary size. Of course, no finite graph P can satisfy the necessary condition—weak predecessor subgraph of predecessor—for all of these weak predecessors.

Theorem 4.5. *Let G be a connected graph. G has no predecessor if and only if G has infinitely many weak predecessors.*

Proof. ($\Leftarrow$) Infinitely many weak predecessors cannot have a bound in size. Thus there is no predecessor as explained.

($\Rightarrow$) Let $W_1, W_2, W_3 \ldots, W_n$ be all the weak predecessors of G and let W be weak $(W_1 \cup W_2 \cup W_3 \cup \cdots \cup W_n, G)$. For any graph $H \trianglelefteq G$ the relation $H \trianglelefteq \text{weak}(H, G) = W_i \trianglelefteq W \trianglelefteq G$ holds and therefore W is a predecessor of G.

5. Density of color-families

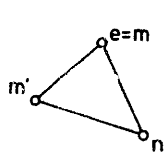
A class $\mathcal{F}$ of families is *dense* if for every pair $\mathcal{L}_1 \subsetneq \mathcal{L}_2$ in $\mathcal{F}$ there is a family $\mathcal{L}_3$ in $\mathcal{F}$ satisfying $\mathcal{L}_1 \subsetneq \mathcal{L}_3 \subsetneq \mathcal{L}_2$. The proof of the density of color-families will be given stepwise. Every step is an expansion of the former step.

Step 1: C_3 has no predecessor. As claimed in Theorem 4.5 we construct infinitely many weak predecessors. For this we shall complete the flowers F_2^n $(n \geq 1)$ to obtain so-called 'super-flowers', which are weak predecessors of C_3. As preparation we discuss some facts about F_2^n flowers.

Let us classify the vertices in F_2^n as follows:
- class N contains only the vertex in the center, which participates in every 'petal' of the flower. We call this vertex *nucleus*.
- class M consists of all neighbors of the nucleus. We call them *middle-vertices*.
- class E, finally, contains the remaining vertices, the *exterior-vertices*, which are not elements of class N or M.

(i) The nucleus is in C_5-connection with every vertex in the flower and therefore every elementary homomorphism identifying the nucleus produces a morphic image with a C_3 (or a loop) as a subgraph.

(ii) Every elementary homomorphism identifying a vertex of E with a vertex of M produces a C_3 as follows: Each $e \in E$ has a $m' \in M$ which connects e with the nucleus n. An identification of e with a vertex $m \in M$ (m not equal m') inserts an edge between $e = m$ and the nucleus n. The C_3



is a subgraph of the morphic image of the flower. From (i) and (ii) we see that the only elementary homomorphisms which produce morphic images which can be strong interpretations of C_3 are those identifying vertices within the set E or within the set

M. When we want to construct a weak predecessor of C_3 we have to avoid this by inserting edges.

For further considerations we draw a flower in a different way: All vertices of M and E are on a circle, the nucleus located outside this circle, the edges to the nucleus indicated by short arrows (see Fig. 1).

A vertex x is a positive neighbor of y—$x = \text{pos}(y)$—if x follows y clockwise on the circle (negative neighbor x—$x = \text{neg}(y)$—is a counter-clockwise neighbor of y); see Fig. 1.

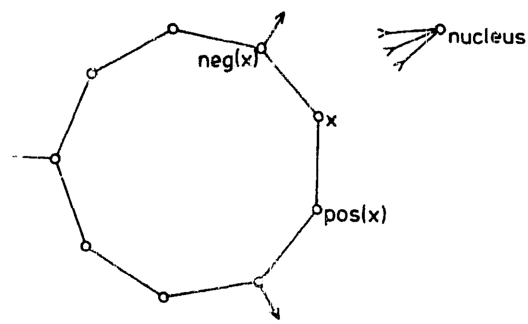


Fig. 1.

Before inserting edges within the circle we have to ensure that our super-flower is still an interpretation of C_3. For this we color the flower with the numbers 1, 2 and 3:
- nucleus 2-colored,
- middle-vertices 1-colored,
- all vertices of E, which are positive neighbors of a middle-vertex 2-colored,
- negative neighbors of middle-vertices 3-colored.

Now we insert edges from every 2-colored vertex of E to every 3-colored vertex of E, except between those 2-3 pairs which are connected by a middle-vertex. We have now obtained the super-flower S_2^n (see Fig. 2).

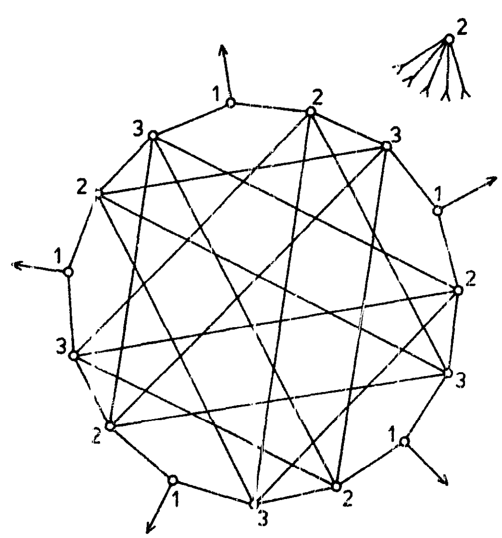


Fig. 2. The super-flower S_2^2.

There are three points left to verify for a super-flower S:

(i) S is an interpretation of C_3,

(ii) C_3 is not an interpretation of S,

(iii) every pair of vertices in S is in C_5-connection.

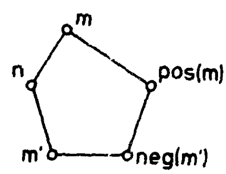
(i) There is no edge between two vertices, colored with the same number. Hence, S is C_3-colorable.

(ii) Suppose there is a C_3 in S. The nucleus cannot be involved in this cycle, because its only neighbors are the middle-vertices which are not adjacent. A middle vertex cannot be in a C_3, because its only neighbors are the nucleus and its positive and negative neighbor, which we did not connect. Consequently, the only possibility is a C_3, consisting only of vertices of E. This would follow one of the number-triples:

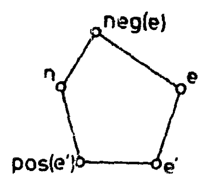
2–2–2, 2–2–3, 2–3–3, 3–3–3.

This is impossible because we did not insert edges between vertices colored with the same number.

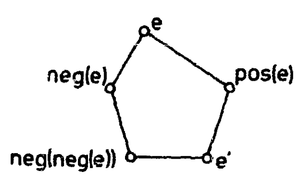
(iii) Two vertices m and m' of M are in C_5-connection:



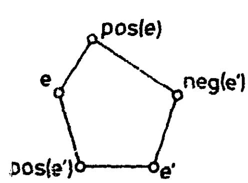
Two vertices e and e' of E are in a common C_5:
–e and e' are adjacent (e.g. e 2-colored and e' 3-colored)



–e and e' are not adjacent (e.g. both 2-colored)

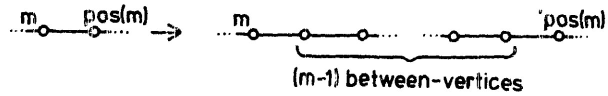


or when pos(e) and e' are connected over a middle-vertex and therefore non-adjacent:



Other cases are already treated in the beginning of this step.

Step 2: C_{2m+1} has no predecessor ($m \geq 1$). Blowing up the super-flowers S_2^n to general super-flowers S_{m+1}^n we shall obtain infinitely many weak predecessors for every C_{2m+1} with weak (S_{m+1}^n, C_{2m+1}). For this we insert a chain of $(m-1)$ vertices instead of the edges between every middle-vertex and its positive and negative neighbors in S_2^n:



By this we create a fourth class of vertices in S_{m+1}^n between M and E, the '*between*'-*class B*.

First we have to verify, that the assumption $S_{m+1}^n \lneqq C_{2m+1}$ for the algorithm weak is satisfied: S_{m+1}^n is C_{2m+1}-colorable as follows (cf. Fig. 3):

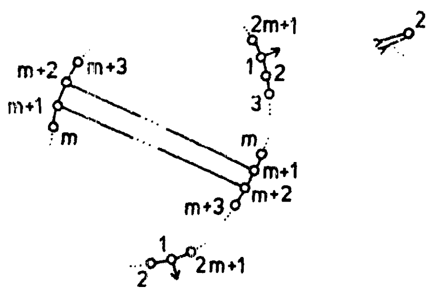


Fig. 3.

- nucleus 2-colored
- middle-vertices 1-colored
- coloring of the between- and exterior-vertices: vertex x (not equal the nucleus) is i-colored ($i = 1, 2, \ldots, 2m$) implies pos(x) is $i+1$ colored.

As no C_{2m+1} is a subgraph of S_{m+1}^n, the relation is strong as desired. Hence, it is valid to speak of $(S_{m+1}^n)' = \text{weak}(S_{m+1}^n, C_{2m+1})$. When two middle-vertices of S_{m+1}^n are identified we get a morphic image S^*, which has a C_{2m+1} as subgraph and (one just has to take the coloring as an identification-instruction) is form equivalent to C_{2m+1}. Therefore, (see remark 4.3.) the resulting homomorphism h of weak (S_{m+1}^n, C_{2m+1}) may not identify the $(2n+1)$ middle-vertices and the number of vertices of $(S_{m+1}^n)'$ exceeds $2n+1$. This implies that we have infinitely many weak predecessors. (Actually it is possible to show that S_{m+1}^n equals $(S_{m+1}^n)'$, but it is not necessary here).

Step 3: Every connected graph G' with og$(G') \neq \infty$ has no predecessor. Let G be the minimal graph of G'. When og$(G) = 2m-1$, mark an aribtrary C_{2m-1} in G. Now we take a S_m^n and build it into G instead of the C_{2m-1} as follows:

- color the C_{2m-1} with the numbers $1, 2 \cdots 2m-1$ cyclically,
- color S_m^n with the numbers $1, 2, \ldots, 2m-1$ as described in the former proof,
- when there was an edge between a vertex x of G (but not of the marked C_{2m-1}) and an i-colored vertex in C_{2m+1}, insert edges between x and all i-colored vertices in S_m^n.

We denote the resulting graph by $G(S_m^n)$. The coloring of S_m^n gives an identification-instruction which results in G—consequently we have $G(S_m^n) \lhd G$. To ensure that $G(S_m^n) \lneqq G$ we show that G is not a subgraph of $G(S_m^n)$ as follows:

Suppose G is a subgraph of $G(S_m^n)$. Then exactly $2m-1$ vertices of S_m^n, all colored with different numbers, are 'involved' in this subgraph G.

Arguments: (i) 'At least $2m-1$ vertices', because otherwise there are not enough vertices for G.

(ii) 'All colored with different numbers', because otherwise our 'identification-instruction' would identify two vertices of this subgraph G. But in the end the morphic image equals G. Hence, the identification-instruction, applied only on the subgraph G in $G(S_m^n)$, would produce a morphic proper subgraph of G that is a contradiction to the minimality of G.

(iii) 'At most $2m-1$ vertices', because S_m^n is colored with only $2m-1$ numbers.

But these $2m-1$ vertices in S_m^n would have to form the formerly replaced C_{2m-1} which is a contradiction because there is no C_{2m-1} in S.

Using the same arguments as in the former step we have the estimate that the number of vertices of weak$(G(S_m^n), G)$ exceeds $2n+1$ and again we have infinitely many weak predecessors.

Step 4: And

Theorem 5.1. *Color-families are dense (excluding the pair $\mathscr{L}(K_1)$, $\mathscr{L}(K_2)$).*

It is only left to show that every graph G with an odd cycle—even not connected—has no predecessor. Of course, this implies density.

Assume $P = \bigcup_{i=1}^k P_i$ is predecessor of $S = \bigcup_{j=1}^l S_j$ (let P and S be minimal graphs, P_i, S_j be connected graphs). Because of minimality we have P_i incomparable with P_j and S_i incomparable with S_j for any pair $i \neq j$. Denote $\mathscr{S} = \{S_j\}_{j=1,\dots,l}$ and $\mathscr{P} = \{P_i\}_{i=1,\dots,k}$.

Case 1: $\mathscr{P} \subset \mathscr{S}$ and $k > l+1$ implies $P \lneqq P \cup R \lneqq S$ where R is a graph of $\mathscr{S} \backslash \mathscr{P}$.

Case 2: $\mathscr{R} \subset \mathscr{S}$ and $k = l+1$. Let R be the graph which is in $\mathscr{S}$ but not in $\mathscr{P}$, let og(R) be $2m-1$ and n be the number of vertices of the largest graph in $\mathscr{P}$. When we construct a graph weak$(R(S_m^n), R)$ according to the proof in Step 3, we have the relation $P \lneqq P \cup \text{weak}(R(S_m^n), R) \lneqq S$.

Case 3: $\mathscr{P} \not\subset \mathscr{S}$. Take a graph Q of $\mathscr{P}$ which is not in $\mathscr{S}$. Then there must be a graph S_i of $\mathscr{S}$ so that Q is a strong interpretation of S_i. Let M be a graph which satisfies $Q \lneqq M \lneqq S$ (there must be such a graph M because we have already shown density for connected graphs). When P' is the graph P without the component Q, the relation $P \lneqq P' \cup M \lneqq S$ implies our final contradiction against $P \cdot \text{p} \cdot S$.

Let us finish this section with a short remark. At first sight the proof given here is not constructive in the sense that it does not describe an algorithm for 'squeezing in' graphs, explicitly. In fact the proof can be 'used constructively' in the following way: Consider a pair of graphs H and G, where G is a strong interpretation of H (H

connected). Let n be the number of vertices of G. Let W be a weak predecessor with more than n vertices. W cannot be an interpretation of G, because the morphic image W^* of W which would necessarily be a subgraph of G would be a strong interpretation of H. So we have either $G \lhd W \lhd H$ or if G and W are incomparable, we have $G \lhd G \cup W \lhd H$. As constructions for weak predecessors were given, the proof is constructive from this point of view.

Acknowledgment

I would like to thank H.A. Maurer, A. Salomaa and D. Wood for suggesting the problem of density and for the help in the formulation.

References

[1] F. Harary, *Graph Theory* (Addison-Wesley Reading, MA, 1972).
[2] H.A. Maurer, A. Salomaa and D. Wood, Colorings and interpretations: A connection between graphs and grammar forms, Report 49, Institut für Informationsverarbeitung, TU Graz (1980).
[3] H.A. Maurer, A. Salomaa and D. Wood, On finite grammar forms, Report 51, Institut für Informationsverarbeitung, TU Graz (1980).
[4] H.A. Maurer, A. Salomaa and D. Wood, On predecessors of finite languages, Report 57, Institut für Informationsverarbeitung, TU Graz (1980).
[5] D. Wood, *Grammar and L Forms: An Introduction* (Springer, Berlin, 1980).